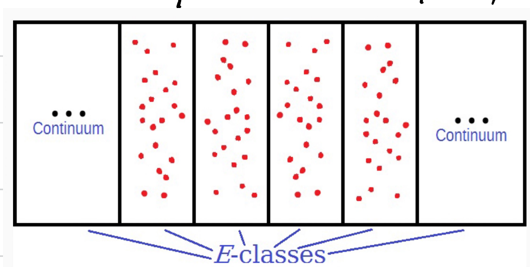
Ergodic Theory and Measured Group Theory
Lecture 22

To get an idea of what measured group theory is, let's discuss the following motivating question.

Question 1. For $n \neq m$, can free groups $\mathbb{F}_n$ and $\mathbb{F}_m$ have free pmp orbit equivalent actions? In other words, for any $n, m \in \mathbb{N} \cup \{\infty\}$, if pmp actions of $\mathbb{F}_n$ and $\mathbb{F}_m$ induce the same orbit eq. rel. on (X, μ), must $n = m$?



More generally, we can ask the following:

Rigidity Question. If an eq. rel. E is induced by a free pmp action of a ctbl group Γ, how much about Γ does E remember?

This is a typical question in measured group theory.

We can ask the same question as Question 1 but for free abelian groups $\mathbb{Z}^n$.

Question 2. For $n \neq m$, $n, m \geq 1$, can $\mathbb{Z}^n$ and $\mathbb{Z}^m$ have orbit equivalent free pmp actions?

Take $\mathbb{Z}$ & $\mathbb{Z}^2$. The orbits of a $\mathbb{Z}$-action are $\mathbb{Z}$-lines, while those of $\mathbb{Z}^2$ are 2D-grids. Question 2 is asking if it's possible to structure a.e. orbit both as a line and a grid in a Borel fashion. Turns out that yes, we can!

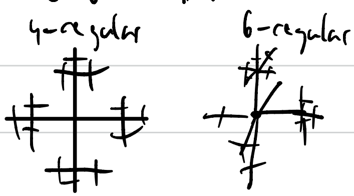
Theorem (Elasticity of amenable groups, Dye for $\mathbb{Z}$-actions, Ornstein–Weiss for all amenable groups). Any ergodic free pmp actions of any two amenable groups are orbit equivalent. In other words, there is only one ergodic CBER induced by a free pmp action of an amenable group.

Remark. A CBER is hyperfinite $\Leftrightarrow$ it's induced by a Borel action of $\mathbb{Z}$. The above theorem says that the orbit eq. rel. induced by a free pmp action of an amenable group can be induced by an action of $\mathbb{Z}$, i.e. is hyperfinite (mod null). Connes, Feldman, Weiss proved that one can drop the pmp assumption to just measurable and the eq. rel. still is hyperfinite mod null.

<u>Open question (Borel).</u> Does any Borel action of an amenable group induce a hyperfinite eq. rel. (no modding with null sets)?

<u>Latest update.</u> True for polycyclic groups (abelian, nilpotent, virtually nilpotent).

Okay so the amenable groups are elastic, but our original question is for free groups (not amenable).
Again, for $\mathbb{F}_2$ and $\mathbb{F}_3$, having the same orbit eq. rel. means that each orbit can be structured by both a 4-regular trees and 6-regular trees. The question is whether it is possible?



Rigidity for free groups (Gaborian 1997). No, it's not possible, i.e. if free pmp actions of $\mathbb{F}_n$ and $\mathbb{F}_m$ are orbit equivalent, then $n=m$.

We'll work towards understanding how this was answered, but let's consider the rigidity for all nonamenable groups.

Theorem (Ioana 2007, Epstein 2008). If Γ is nonamenable, then it admits continuum-many non-orbit-equiv. ergodic free pmp actions.

$\forall \Gamma \geq \mathbb{F}_2$ (Ioana 2007); $\forall$ nonamenable (Epstein 2008)

Ioana's result doesn't apply to general nonamenable groups because they may not contain $\mathbb{F}_2$. However, one instead use the following probabilistic solution to the Day-von Neumann question:

Theorem (Gaborian-Lyons). If Γ is nonamenable, then the orbit eq. rel. E_Γ of the Bernoulli action $\Gamma \curvearrowright (\{0,1\}^\Gamma, \lambda^\Gamma)$ contains a subequivalence relation $E_{\mathbb{F}_2}$ induced by a free pmp action of $\mathbb{F}_2$.

+ a very clever construction

This is what Epstein used to deduce the result for all nonamenable groups from Ioana's theorem for $\Gamma \geq \mathbb{F}_2$.

Graphings and cost.

We go back to rigidity question for free groups.
So why $\mathbb{F}_2$ and $\mathbb{F}_3$ are not isomorphic? There are many ways to see that $\mathbb{F}_3$ cannot be generated by <3 generators, i.e. $\mathrm{rank}(\mathbb{F}_2) = 2 \neq 3 = \mathrm{rank}(\mathbb{F}_3)$.

Def. For a ctbl gp Γ, $\mathrm{rank}(\Gamma)$ is the minimum number of generators needed to generate Γ.

The idea is to adapt the notion of rank to orbit eq. relations. To do so let's view the rank as minimum over all Cayley graphs of Γ of $\frac{1}{2}$ of the degree of each vertex. So we adapt the notion of Cayley graphs to eq. rel.

Def. A Borel graph G on a ct. Borel space X is just a

Borel subset of X^2 that is irreflexive (i.e. $(x,x) \notin G$) and (undirected $\Leftrightarrow$ symmetric). Let E_G denote the connectedness eq. rel. If G is locally ctbl (i.e. each vertex has only ctbly-many neighbours), then easy to see that E_G is a CBER.

- Converseley, given a CBER E on X, we call a Borel graph a graphing of E if $E_G = E$.
  "Cayley graph"